Termination of lambda-calculus with the extra Call-By-Value rule known as assoc.

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Notations and standard results are presented in Appendix A.

We consider the following rule in λ-calculus:

$$\mathsf{assoc} \quad (\lambda x.M)\ ((\lambda y.N)\ P) \longrightarrow\ (\lambda y.(\lambda x.M)\ N)\ P)$$

We want to prove

Proposition 1 $\mathit{SN}^{\beta} \subseteq \mathit{SN}^{\mathit{assoc}\beta}$.

Lemma 1 $\longrightarrow_{\mathit{assoc}}$ *is terminating in λ-calculus.*

Proof: Each application of the rule decreases by one the number of pairs of λ that are not nested. □

To prove Proposition 1 above, it would thus be sufficient to prove that $\longrightarrow_{\mathsf{assoc}}$ could be adjourned with respect to $\longrightarrow_{\beta}$, in other words that $\longrightarrow_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}\ \subseteq\ \longrightarrow_{\beta} \cdot \longrightarrow^{*}_{\mathsf{assoc}\beta}$ (the adjournment technique leads directly to the desired strong normalisation result). When trying to prove the property by induction and case analysis on the β-reduction following the assoc-reduction to be adjourned, all cases allow the adjournment but one, namely:

$$(\lambda x.M)\ ((\lambda y.N)\ P) \longrightarrow_{\mathsf{assoc}} (\lambda y.(\lambda x.M)\ N)\ P \longrightarrow_{\beta} (\lambda y.\left\{{}^{N}\!/\!_{x}\right\}M)\ P$$

Hence, we shall assume without loss of generality that the β-reduction is not of the above kind. For that we need to identify a sub-relation of β-reduction $\hookrightarrow$ such that

- $\longrightarrow_{\mathsf{assoc}}$ can now be adjourned with respect to $\hookrightarrow$
- we can justify that there is no loss of generality.

For this we give ourselves the possibility of marking λ-redexes and forbid reductions under their (marked) bindings, so that, if in the assoc-reduction above we make sure that $(\lambda y.(\lambda x.M)\ N)\ P)$ is marked, the problematic β-reduction is forbidden.

Hence we use the usual notation for a marked redex $(\overline{\lambda} y.Q)\ P$, but we can also see it as the construct $\mathsf{let}\ y = P\ \mathsf{in}\ Q$ of λ_{C} [Mog88] and other works on call-by-value λ-calculus. We start with a reminder about marked redexes.

Definition 1 The syntax of the λ-calculus is extended as follows:

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid (\overline{\lambda} x.M)\ N$$

Reduction is given by the following system $\beta 12$:

$$\begin{array}{lll} \beta 1 & (\lambda x.M)\ N & \longrightarrow \left\{ {}^{M}\!/\!{}_{x} \right\} N \\ \beta 2 & (\overline{\lambda} x.M)\ N & \longrightarrow \left\{ {}^{M}\!/\!{}_{x} \right\} N \end{array}$$

The forgetful projection onto λ-calculus is straightforward:

$$\begin{array}{lll} \phi(x) & := & x \\ \phi(\lambda x.M) & := & \lambda x.\phi(M) \\ \phi(M\ N) & := & \phi(M)\ \phi(N) \\ \phi((\overline{\lambda} x.M)\ N) & := & (\lambda x.\phi(M))\ \phi(N) \end{array}$$

Remark 2 Clearly, $\longrightarrow_{\beta 12}$ strongly simulates $\longrightarrow_{\beta}$ through ϕ^{-1} and $\longrightarrow_{\beta}$ strongly simulates $\longrightarrow_{\beta 12}$ through ϕ.

Reducing under $\overline{\lambda}$ and erasing $\overline{\lambda}$ can be strongly adjourned

In this section we identify the reduction notion $\hookrightarrow$ $(\subseteq \longrightarrow_{\beta 12})$ and we argue against the loss of generality by proving that $\longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot (\longrightarrow_{\beta 12} \cup \hookrightarrow)^+$, a strong case of adjournment, presented in Appendix B, whose direct corollary is that, for every sequence of $\beta 12$-reduction, there is also a sequence of $\hookrightarrow$-reduction of the same length and starting from the same term.

We thus split the reduction system $\beta 12$ into two cases depending on whether or not a reduction throws away an argument that contains some markings:

Definition 2

$$\begin{array}{ll} \beta\kappa & \left\{ \begin{array}{lll} (\lambda x.M)\ P & \longrightarrow\ M & \text{if } x \notin \mathsf{FV}(M) \text{ and there is a term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \\ (\overline{\lambda} x.M)\ P & \longrightarrow\ M & \text{if } x \notin \mathsf{FV}(M) \text{ and there is a term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \end{array} \right. \\ \beta\overline{\kappa} & \left\{ \begin{array}{lll} (\lambda x.M)\ P & \longrightarrow\ M & \text{if } x \in \mathsf{FV}(M) \text{ or there is no term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \\ (\overline{\lambda} x.M)\ P & \longrightarrow\ M & \text{if } x \in \mathsf{FV}(M) \text{ or there is no term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \end{array} \right. \end{array}$$

Remark 3 Clearly, $\longrightarrow_{\beta 12} = \longrightarrow_{\beta\kappa} \cup \longrightarrow_{\beta\overline{\kappa}}$.

No we distinguish whether or not a reduction occurs underneath a marked redex, via the following rule and the following notion of contextual closure:

Definition 3

$$\overline{\beta} \quad (\overline{\lambda} x.M)\ P \ \longrightarrow\ (\overline{\lambda} x.N)\ P \quad \text{if } M \longrightarrow_{\beta 12} N$$

Now we define a weak notion of contextual closure for a rewriting system i:

$$\frac{i:\ M \longrightarrow\ N}{M \rightharpoonup_i N} \qquad \frac{M \rightharpoonup_i N}{\lambda x.M \rightharpoonup_i \lambda x.N} \qquad \frac{M \rightharpoonup_i N}{M\ P \rightharpoonup_i N\ P} \qquad \frac{M \rightharpoonup_i N}{P\ M \rightharpoonup_i P\ N}$$

$$\frac{M \rightharpoonup_i N}{(\overline{\lambda} x.P)\ M \rightharpoonup_i (\overline{\lambda} x.P)\ N}$$

Finally we use the following abbreviations:

Definition 4 Let $\hookrightarrow := \rightharpoonup_{\beta\overline{\kappa}}$ and $\leadsto_1 := \rightharpoonup_{\beta\kappa}$ and $\leadsto_2 := \rightharpoonup_{\overline{\beta}}$.

Remark 4 Clearly, $\longrightarrow_{\beta 12} = \hookrightarrow \cup \leadsto_1 \cup \leadsto_2$.

Lemma 5 *If $(\overline{\lambda} x.N)\ Q \sqsubseteq P$, then there is P' such that $P \hookrightarrow P'$.*

Proof: By induction on P

- The case $P = y$ is vacuous.
- For $P = \lambda y.M$, we have $(\overline{\lambda}x.N)\ Q \sqsubseteq M$ and the induction hypothesis provides $M \hookrightarrow M'$, so $\lambda y.M \hookrightarrow \lambda y.M'$.
- For $P = M_1\ M_2$, we have either $(\overline{\lambda}x.N)\ Q \sqsubseteq M_1$ or $(\overline{\lambda}x.N)\ Q \sqsubseteq M_2$. In the former case the induction hypothesis provides $M_1 \hookrightarrow M_1'$, so $M_1\ M_2 \hookrightarrow M_1'\ M_2$. The latter case is similar.
- Suppose $P = (\overline{\lambda}y.M_1)\ M_2$. If there is a term $(\overline{\lambda}x'.N')\ Q' \sqsubseteq M_2$, the induction hypothesis provides $M_2 \hookrightarrow M_2'$, so $(\overline{\lambda}y.M_1)\ M_2 \hookrightarrow (\overline{\lambda}y.M_1)\ M_2'$. If there is no such term $(\overline{\lambda}x'.N')\ Q' \sqsubseteq M_2$, we have $(\overline{\lambda}y.M_1)\ M_2 \hookrightarrow \left\{{}^{M_2}\!/_{y}\right\} M_1$.

□

Lemma 6 $\rightsquigarrow_1 \subseteq \hookrightarrow \cdot \rightsquigarrow_1$

Proof: By induction on the reduction step $\rightsquigarrow_1$.

For the base cases $(\lambda x.M)\ P \longrightarrow_{\beta\kappa} M$ or $(\overline{\lambda}x.M)\ P \longrightarrow_{\beta\kappa} M$ with $x \notin \mathsf{FV}(M)$ and $(\overline{\lambda}y.N)\ Q \sqsubseteq P$, Lemma 5 provides the reduction $P \hookrightarrow P'$, so $(\lambda x.M)\ P \hookrightarrow (\lambda x.M)\ P' \rightsquigarrow_1 M$ and $(\overline{\lambda}x.M)\ P \hookrightarrow (\overline{\lambda}x.M)\ P' \rightsquigarrow_1 M$.

The induction step is straightforward as the same contextual closure is used on both sides (namely, the weak one). □

Lemma 7 $\rightsquigarrow_2 \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12}$

Proof: By induction on the reduction step $\hookrightarrow$. See appendix C. □

Corollary 8 $\longrightarrow_{\beta 12}$ *can be strongly adjourned with respect to* $\hookrightarrow$.

Proof: Straightforward from the last two theorems, and Remark 4. □

assoc-reduction

We introduce two new rules in the marked λ-calculus to simulate assoc:

$$\begin{array}{lll} \overline{\mathsf{assoc}} & (\overline{\lambda}x.M)\ (\overline{\lambda}y.N)\ P & \longrightarrow\ (\overline{\lambda}y.(\overline{\lambda}x.M)\ N)\ P \\ \mathsf{act} & (\lambda x.M)\ N & \longrightarrow\ (\overline{\lambda}x.M)\ N \end{array}$$

Remark 9 Clearly, $\longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}}$ strongly simulates $\longrightarrow_{\mathsf{assoc}}$ through ϕ^{-1}.

Notice that with the $\mathsf{let} = \mathsf{in}$ -notation, $\overline{\mathsf{assoc}}$ and act are simply the rules of λ_{C}

$$\begin{array}{lll} \overline{\mathsf{assoc}} & \mathsf{let}\ x = (\mathsf{let}\ y = P\ \mathsf{in}\ N)\ \mathsf{in}\ M & \longrightarrow\ \mathsf{let}\ y = P\ \mathsf{in}\ \mathsf{let}\ x = N\ \mathsf{in}\ M \\ \mathsf{act} & (\lambda x.M)\ N & \longrightarrow\ \mathsf{let}\ x = N\ \mathsf{in}\ M \end{array}$$

Lemma 10 $\longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}}$

Proof: By induction on the reduction step $\hookrightarrow$. See appendix C. □

Lemma 11 $\longrightarrow^*_{\overline{assoc},act} \cdot \longrightarrow_{\beta 12}$ *can be strongly adjourned with respect to* $\hookrightarrow$.

Proof: We prove that $\forall k, \longrightarrow^k_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$ by induction on k.

- For $k = 0$, this is Corollary 8.
- Suppose it is true for k. By the induction hypthesis we get

  $$\longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow^k_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

  Then by Lemma 10 we get

  $$\longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \subseteq \hookrightarrow \cdot \longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

□

Remark 12 Note from Lemma 5 that $\mathsf{nf}^{\hookrightarrow} \subseteq \mathsf{nf}^{\rightsquigarrow_1 \cup \rightsquigarrow_2} \subseteq \mathsf{nf}^{\longrightarrow_{\beta 12}} \subseteq \mathsf{nf}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$.

Theorem 13 $BN^{\hookrightarrow} \subseteq BN^{\longrightarrow^*_{\overline{assoc},act} \cdot \longrightarrow_{\beta 12}}$

Proof: We apply Theorem 28, since $\mathsf{nf}^{\hookrightarrow} \subseteq \mathsf{nf}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$ and clearly

$$(\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}) \cup \hookrightarrow = \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

□

Theorem 14 $BN^{\beta} \subseteq BN^{\longrightarrow^*_{assoc} \cdot \longrightarrow_{\beta}}$

Proof: Since $\longrightarrow_\beta$ strongly simulates $\hookrightarrow$ through ϕ, we have $\phi^{-1}(\mathsf{BN}^\beta) \subseteq \mathsf{BN}^{\hookrightarrow} \subseteq \mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$. Hence $\phi(\phi^{-1}(\mathsf{BN}^\beta)) \subseteq \phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}})$. Since ϕ is surjective, $\mathsf{BN}^\beta = \phi(\phi^{-1}(\mathsf{BN}^\beta))$. Hence $\mathsf{BN}^\beta \subseteq \phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}})$. Also, $\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$ strongly simulates $\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta$ through ϕ^{-1}, so $\phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}) \subseteq \mathsf{BN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta}$. □

Theorem 15 $SN^{\beta} \subseteq SN^{assoc\beta}$

Proof: First, from Lemma 19, $\mathsf{BN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta} \subseteq \mathsf{SN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta}$. Then from Lemma 1, $\longrightarrow_{\mathsf{assoc}}$ is terminating and hence $\mathsf{SN}^{\mathsf{assoc}}$ is stable under $\longrightarrow_\beta$. Hence we can apply Lemma 24 to get $\mathsf{SN}^{\mathsf{assoc}\beta} = \mathsf{SN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta}$. From the previous theorem we thus have $\mathsf{BN}^\beta \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$. Now, noticing that β-reduction in λ-calculus is finitely branching, Lemma 18 gives $\mathsf{BN}^\beta = \mathsf{SN}^\beta$ and thus $\mathsf{SN}^\beta \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$. □

References

[Mog88] E. Moggi. Computational lambda-calculus and monads. Report ECS-LFCS-88-66, University of Edinburgh, Edinburgh, Scotland, October 1988.

A Reminder: Notations, Definitions and Basic Results

Definition 5 (Relations)

- We denote the composition of relations by $\cdot$, the identity relation by Id, and the inverse of a relation by $^{-1}$.
- If $\mathcal{D} \subseteq \mathcal{A}$, we write $\mathcal{R}(\mathcal{D})$ for $\{M \in \mathcal{B} | \exists N \in \mathcal{D}, N\mathcal{R}M\}$, or equivalently $\bigcup_{N\in\mathcal{D}}\{M \in \mathcal{B} | N\mathcal{R}M\}$. When $\mathcal{D}$ is the singleton $\{M\}$, we write $\mathcal{R}(M)$ for $\mathcal{R}(\{M\})$.
- We say that a relation $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ is *total* if $\mathcal{R}^{-1}(\mathcal{B}) = \mathcal{A}$.

Remark 16 Composition is associative, and identity relations are neutral for the composition operation.

Definition 6 (Reduction relation)

- A *reduction relation* on $\mathcal{A}$ is a relation from $\mathcal{A}$ to $\mathcal{A}$.
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we define the set of $\rightarrow$-*reducible forms* (or just *reducible forms* when the relation is clear) as $\mathsf{rf}^{\rightarrow} := \{M \in \mathcal{A} | \exists N \in \mathcal{A}, M \rightarrow N\}$. We define the set of *normal forms* as $\mathsf{nf}^{\rightarrow} := \{M \in \mathcal{A} | \not\exists N \in \mathcal{A}, M \rightarrow N\}$.
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we write $\leftarrow$ for $\rightarrow^{-1}$, and we define $\rightarrow^n$ by induction on the natural number n as follows:
  $\rightarrow^0 := \mathsf{Id}$
  $\rightarrow^{n+1} := \rightarrow \cdot \rightarrow^n (= \rightarrow^n \cdot \rightarrow)$
  $\rightarrow^+$ denotes the transitive closure of $\rightarrow$ (i.e. $\rightarrow^+ := \bigcup_{n\geq 1} \rightarrow^n$).
  $\rightarrow^*$ denotes the transitive and reflexive closure of $\rightarrow$ (i.e. $\rightarrow^* := \bigcup_{n\geq 0} \rightarrow^n$).
  $\leftrightarrow$ denotes the symmetric closure of $\rightarrow$ (i.e. $\leftrightarrow := \leftarrow \cup \rightarrow$).
  $\leftrightarrow^*$ denotes the transitive, reflexive and symmetric closure of $\rightarrow$.
- An *equivalence relation* on $\mathcal{A}$ is a transitive, reflexive and symmetric reduction relation on $\mathcal{A}$, i.e. a relation $\rightarrow = \leftrightarrow^*$, hence denoted more often by $\sim$, $\equiv$...
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$ and a subset $\mathcal{B} \subseteq \mathcal{A}$, the *closure of* $\mathcal{B}$ *under* $\rightarrow$ is $\rightarrow^*(\mathcal{B})$.

Definition 7 (Finitely branching relation) A reduction relation $\rightarrow$ on $\mathcal{A}$ is *finitely branching* if $\forall M \in \mathcal{A}$, $\rightarrow(M)$ is finite.

Definition 8 (Stability) Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we say that a subset $\mathcal{T}$ of $\mathcal{A}$ is $\rightarrow$-*stable* (or *stable under* $\rightarrow$) if $\rightarrow(\mathcal{T}) \subseteq \mathcal{T}$.

Definition 9 (Strong simulation)
Let $\mathcal{R}$ be a relation between two sets $\mathcal{A}$ and $\mathcal{B}$, respectively equipped with the reduction relations $\rightarrow_\mathcal{A}$ and $\rightarrow_\mathcal{B}$.
$\rightarrow_\mathcal{B}$ *strongly simulates* $\rightarrow_\mathcal{A}$ *through* $\mathcal{R}$ if $(\mathcal{R}^{-1} \cdot \rightarrow_\mathcal{A}) \subseteq (\rightarrow_\mathcal{B}^+ \cdot \mathcal{R}^{-1})$.

Remark 17

1. If $\rightarrow_\mathcal{B}$ strongly simulates $\rightarrow_\mathcal{A}$ through $\mathcal{R}$, and if $\rightarrow_\mathcal{B} \subseteq \rightarrow'_\mathcal{B}$ and $\rightarrow'_\mathcal{A} \subseteq \rightarrow_\mathcal{A}$, then $\rightarrow'_\mathcal{B}$ strongly simulates $\rightarrow'_\mathcal{A}$ through $\mathcal{R}$.

2. If $\to_{\mathcal{B}}$ strongly simulates $\to_{\mathcal{A}}$ and $\to'_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly simulates $\to_{\mathcal{A}} \cdot \to'_{\mathcal{A}}$ through $\mathcal{R}$.

3. Hence, if $\to_{\mathcal{B}}$ strongly simulates $\to_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly simulates $\to^{+}_{\mathcal{A}}$ through $\mathcal{R}$.

Definition 10 (Patriarchal) Given a reduction relation $\to$ on $\mathcal{A}$, we say that

- a subset $\mathcal{T}$ of $\mathcal{A}$ is $\to$-*patriarchal* (or just *patriarchal* when the relation is clear) if $\forall N \in \mathcal{A},\ \to(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$.
- a predicate P on $\mathcal{A}$ is *patriarchal* if $\{M \in \mathcal{A} |\ P(M)\}$ is *patriarchal*.

Definition 11 (Normalising elements) Given a reduction relation $\to$ on $\mathcal{A}$, the set of $\to$-*strongly normalising* elements is

$$\mathsf{SN}^{\to} := \bigcap_{\mathcal{T} \text{ is patriarchal}} \mathcal{T}$$

Definition 12 (Bounded elements) The set of $\to$-*bounded* elements is defined as

$$\mathsf{BN}^{\to} := \bigcup_{n \geq 0} \mathsf{BN}^{\to}_{n}$$

where $\mathsf{BN}^{\to}_{n}$ is defined by induction on the natural number n as follows:

$$\begin{array}{lll} \mathsf{BN}^{\to}_{0} & := & \mathsf{nf}^{\to} \\ \mathsf{BN}^{\to}_{n+1} & := & \{M \in \mathcal{A} |\ \exists n' \leq n,\ \to(M) \subseteq \mathsf{BN}^{\to}_{n'}\} \end{array}$$

Lemma 18 *If $\to$ is finitely branching, then $\mathsf{BN}^{\to}$ is patriarchal.*
As a consequence, $\mathsf{BN}^{\to} = \mathsf{SN}^{\to}$.

Lemma 19

1. *If $n < n'$ then $\mathsf{BN}^{\to}_{n} \subseteq \mathsf{BN}^{\to}_{n'} \subseteq \mathsf{BN}^{\to}$. In particular, $\mathsf{nf}^{\to} \subseteq \mathsf{BN}^{\to}_{n} \subseteq \mathsf{BN}^{\to}$.*
2. *$\mathsf{BN}^{\to} \subseteq \mathsf{SN}^{\to}$.*

Lemma 20

1. *$\mathsf{SN}^{\to}$ is patriarchal.*
2. *If $M \in \mathsf{BN}^{\to}$ then $\to(M) \subseteq \mathsf{BN}^{\to}$.*
   If $M \in \mathsf{SN}^{\to}$ then $\to(M) \subseteq \mathsf{SN}^{\to}$.

Theorem 21 (Induction principle) *Given a predicate P on $\mathcal{A}$,*
suppose $\forall M \in \mathsf{SN}^{\to}, (\forall N \in \to(M), P(N)) \Rightarrow P(M)$.
Then $\forall M \in \mathsf{SN}^{\to}, P(M)$.

When we use this theorem to prove a statement $P(M)$ for all M in $\mathsf{SN}^{\to}$, we just add $(\forall N \in \to(M), P(N))$ to the assumptions, which we call the induction hypothesis.

We say that we prove the statement by induction in $\mathsf{SN}^{\to}$.

Lemma 22

1. *If $\to_1 \subseteq \to_2$, then $\mathsf{nf}^{\to_1} \supseteq \mathsf{nf}^{\to_2}$, $\mathsf{SN}^{\to_1} \supseteq \mathsf{SN}^{\to_2}$,*
   and for all n, $\mathsf{BN}^{\to_1}_{n} \supseteq \mathsf{BN}^{\to_2}_{n}$.

2. $\mathsf{nf}^{\rightarrow} = \mathsf{nf}^{\rightarrow^+}$, $\mathsf{SN}^{\rightarrow} = \mathsf{SN}^{\rightarrow^+}$, *and for all* n, $\mathsf{BN}_n^{\rightarrow^+} = \mathsf{BN}_n^{\rightarrow}$.

Notice that this result enables us to use a stronger induction principle: in order to prove $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$, it now suffices to prove

$$\forall M \in \mathsf{SN}^{\rightarrow}, (\forall N \in \rightarrow^{+}(M), P(N)) \Rightarrow P(M)$$

This induction principle is called the *transitive induction in* $\mathsf{SN}^{\rightarrow}$.

Theorem 23 (Strong normalisation by strong simulation) *Let $\mathcal{R}$ be a relation between $\mathcal{A}$ and $\mathcal{B}$, equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then $\mathcal{R}^{-1}(\mathsf{SN}^{\rightarrow_{\mathcal{B}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathcal{A}}}$.

Lemma 24 *Given two reduction relations $\rightarrow_1$, $\rightarrow_2$, suppose that $\mathsf{SN}^{\rightarrow_1}$ is stable under $\rightarrow_2$. Then $\mathsf{SN}^{\rightarrow_1 \cup \rightarrow_2} = \mathsf{SN}^{\rightarrow_1^* \cdot \rightarrow_2} \cap \mathsf{SN}^{\rightarrow_1}$.*

B Strong adjournment

Definition 13 Suppose $\rightarrow_{\mathcal{A}}$ is a reduction relation on $\mathcal{A}$, $\rightarrow_{\mathcal{B}}$ is a reduction relation on $\mathcal{B}$, $\mathcal{R}$ is a relation from $\mathcal{A}$ to $\mathcal{B}$.

$\rightarrow_{\mathcal{B}}$ *simulates the reduction lengths of* $\rightarrow_{\mathcal{A}}$ *through* $\mathcal{R}$ if

$$\forall k, \forall M, N \in \mathcal{A}, \forall P \in \mathcal{B},\ M \rightarrow_{\mathcal{A}}^{k} N \wedge M\mathcal{R}P \ \Rightarrow\ \exists Q \in \mathcal{B}, P \rightarrow_{\mathcal{B}}^{k} Q$$

Lemma 25 *Suppose $\rightarrow_{\mathcal{A}}$ is a reduction relation on $\mathcal{A}$, $\rightarrow_{\mathcal{B}}$ is a reduction relation on $\mathcal{B}$, $\mathcal{R}$ is a relation from $\mathcal{A}$ to $\mathcal{B}$.*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then $\rightarrow_{\mathcal{B}}$ simulates the reduction lengths of $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$.

Proof: We prove by induction on k that $\forall k, \forall M, N \in \mathcal{A}^2, \forall P \in \mathcal{B},\ M \rightarrow_{\mathcal{A}}^{k} N \wedge M\mathcal{R}P \ \Rightarrow\ \exists Q, P \rightarrow_{\mathcal{B}}^{k} Q$.

- For $k = 0$: take $Q := M = N$.
- Suppose it is true for k and take $M \rightarrow_{\mathcal{A}} M' \rightarrow_{\mathcal{A}}^{k} N$. The strong simulation gives P' such that $P \rightarrow_{\mathcal{B}}^{+} P'$ and $M'\mathcal{R}P'$. The induction hypothesis gives Q' such that $P' \rightarrow_{\mathcal{B}}^{k} Q'$. Then it suffices to take the prefix $P \rightarrow_{\mathcal{B}}^{k+1} Q$ (of length $k+1$) of $P \rightarrow_{\mathcal{B}}^{+} P' \rightarrow_{\mathcal{B}}^{k} Q'$.

□

Lemma 26 $\forall n, \forall M, \quad (\forall k, \forall N, M \rightarrow^{k} N \Rightarrow k \leq n) \quad \Longleftrightarrow \quad M \in \mathsf{BN}_n^{\rightarrow}$

Proof: By transitive induction on n.

- For $n = 0$: clearly both sides are equivalent to $M \in \mathsf{nf}^{\rightarrow}$.
- Suppose it is true for all $i \leq n$.

  Suppose $\forall k, \forall N, M \rightarrow^{k} N \Rightarrow k \leq n+1$. Then take $M \rightarrow M'$ and assume $M' \rightarrow^{k'} N'$. We have $M \rightarrow^{k'+1} N'$ so from the hypothesis we derive $k' + 1 \leq n + 1$, i.e. $k' \leq n$. We apply the induction hypothesis on M' and get $M' \in \mathsf{BN}_n^{\rightarrow}$. By definition of $\mathsf{BN}_{n+1}^{\rightarrow}$ we get $M \in \mathsf{BN}_{n+1}^{\rightarrow}$.

  Conversely, suppose $M \in \mathsf{BN}_{n+1}^{\rightarrow}$ and $M \rightarrow^{k} N$. We must prove that $k \leq n+1$. If $k = 0$ we are done. If $k = k' + 1$ we have $M \rightarrow M' \rightarrow^{k'} N$; by definition of $\mathsf{BN}_{n+1}^{\rightarrow}$ there is $i \leq n$ such that $M' \in \mathsf{BN}_i^{\rightarrow}$, and by induction hypothesis we have $k' \leq i$; hence $k = k' + 1 \leq i + 1 \leq n + 1$.

□

Theorem 27 *Suppose $\to_{\mathcal{A}}$ is a reduction relation on $\mathcal{A}$, $\to_{\mathcal{B}}$ is a reduction relation on $\mathcal{B}$, $\mathcal{R}$ is a relation from $\mathcal{A}$ to $\mathcal{B}$.*

If $\to_{\mathcal{B}}$ simulates the reduction lengths of $\to_{\mathcal{A}}$ through $\mathcal{R}$, then

$$\forall n, \mathcal{R}^{-1}(\mathsf{BN}_n^{\to_{\mathcal{B}}}) \subseteq \mathsf{BN}_n^{\to_{\mathcal{A}}} \quad (\subseteq \mathsf{SN}^{\to_{\mathcal{A}}})$$

Proof: Suppose $N \in \mathsf{BN}_n^{\to_{\mathcal{B}}}$ and $M\mathcal{R}N$. If $M \to_{\mathcal{A}}^k M'$ then by simulation $N \to_{\mathcal{B}}^k N'$ so by Lemma 26 we have $k \leq n$. Hence by (the other direction of) Lemma 26 we have $M \in \mathsf{BN}_n^{\to_{\mathcal{A}}}$. □

Definition 14 Let $\to_1$ and $\to_2$ be two reduction relations on $\mathcal{A}$.
The relation $\to_1$ can be *strongly adjourned with respect to* $\to_2$ if whenever $M \to_1 N \to_2 P$ there exists Q such that $M \to_2 Q(\to_1 \cup \to_2)^+ P$.

Theorem 28 *Let $\to_1$ and $\to_2$ be two reduction relations on $\mathcal{A}$. If $\mathsf{nf}^{\to_2} \subseteq \mathsf{nf}^{\to_1}$ and $\to_1$ can be strongly adjourned with respect to $\to_2$ then $\mathsf{BN}^{\to_2} \subseteq \mathsf{BN}^{\to_1 \cup \to_2}$.*

Proof: From Theorem 27, it suffices to show that $\to_2$ simulates the reduction lengths of $\to_1 \cup \to_2$ through the identity. We show by induction on k that

$$\forall k, \forall M, N,\ M(\to_1 \cup \to_2)^k N \Rightarrow\ \exists Q, M \to_2^k Q$$

- For $k = 0$: take $Q :=\ M$
- For $k = 1$: If $M \to_2 N$ take $Q :=\ N$; if $M \to_1 N$ use the hypothesis $\mathsf{nf}^{\to_2} \subseteq \mathsf{nf}^{\to_1}$ to produce Q such that $M \to_2 Q$.
- Suppose it is true for $k+1$ and take $M(\to_1 \cup \to_2)P(\to_1 \cup \to_2)^{k+1}N$.

  The induction hypothesis provides T such that $P \to_2^{k+1} T$, in other words $P \to_2 S \to_2^k T$.

  If $M \to_2 P$ we are done. If $M \to_1 P$ we use the hypothesis of adjournment to transform $M \to_1 P \to_2 S$ into $M \to_2 P'(\to_1 \cup \to_2)^+ S$. Take the prefix $P'(\to_1 \cup \to_2)^{k+1} R$ (of length $k+1$) of $P'(\to_1 \cup \to_2)^+ S \to_2^k T$, and apply on this prefix the induction hypothesis to get $P' \to_2^{k+1} R$. We thus get $M \to_2^{k+2} R$.

□

C Proofs

Lemma 7 $\rightsquigarrow_2 \cdot \hookrightarrow\ \subseteq\ \hookrightarrow \cdot \longrightarrow_{\beta 12}^+$

Proof: By induction on the reduction step $\hookrightarrow$.

- For the base case where the $\beta\overline{\kappa}$-reduction is a $\beta 2$-reduction, we have $M \rightsquigarrow_2 (\overline{\lambda}x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\overline{\lambda}x.N)\ P$.

  If $M = (\overline{\lambda}x.N')\ P \rightsquigarrow_2 (\overline{\lambda}x.N)\ P$ because $N' \longrightarrow_{\beta 12} N$ then $(\overline{\lambda}x.N')\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N' \longrightarrow_{\beta 12} \left\{{}^{P}\!/\!_{x}\right\}N$.

  If $M = (\overline{\lambda}x.N)\ P' \rightsquigarrow_2 (\overline{\lambda}x.N)\ P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \mathsf{FV}(N)$. Hence $(\overline{\lambda}x.N)\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\}N \longrightarrow_{\beta 12}^+ \left\{{}^{P}\!/\!_{x}\right\}N$.

- For the base case where the $\beta\overline{\kappa}$-reduction is a $\beta 1$-reduction, we have $M \rightsquigarrow_2 (\lambda x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\lambda x.N)\ P$.

  If $M = M'\ P \rightsquigarrow_2 (\lambda x.N)\ P$ because $M' \rightsquigarrow_2 \lambda x.N$ then M' must be of the form $\lambda x.M''$ with $M'' \rightsquigarrow_2 N$. Then $(\lambda x.M'')\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}M''$ (in case P has a marked subterm, notice that $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(M'')$), and $\left\{{}^{P}\!/\!_{x}\right\}M'' \longrightarrow_{\beta 12} \left\{{}^{P}\!/\!_{x}\right\}N$.

  If $M = (\lambda x.N)\ P' \rightsquigarrow_2 (\lambda x.N)\ P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \mathsf{FV}(N)$. Hence $(\lambda x.N)\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\}N \longrightarrow^{+}_{\beta 12} \left\{{}^{P}\!/\!_{x}\right\}N$.

- The closure under λ is straightforward.

- For the closure under application, left-hand side, we have $M \rightsquigarrow_2 N\ P \hookrightarrow N'\ P$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N\ P$.

  If $M = M'\ P \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12} N'$ and the weak contextual closure gives $M'\ P \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12} N'\ P$.

  If $M = N\ P' \rightsquigarrow_2 N\ P$ with $P' \rightsquigarrow_2 P$, we can also derive $N\ P' \hookrightarrow N'\ P' \longrightarrow_{\beta 12} N'\ P$.

- For the closure under application, right-hand side, we have $M \rightsquigarrow_2 N\ P \hookrightarrow N\ P'$ with $P \hookrightarrow P'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N\ P$.

  If $M = M'\ P \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 N$, we can also derive $M'\ P \hookrightarrow M'\ P' \longrightarrow_{\beta 12} N\ P'$.

  If $M = N\ M' \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 P$, the induction hypothesis gives $M' \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12} P'$ and the weak contextual closure gives $N\ M' \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12} N\ P'$.

- For the closure under marked redex we have $M \rightsquigarrow_2 (\overline{\lambda}x.P)\ N \hookrightarrow (\overline{\lambda}x.P)\ N'$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\overline{\lambda}x.P)\ N$.

  If $M = (\overline{\lambda}x.P')\ N \rightsquigarrow_2 (\overline{\lambda}x.P)\ N$ because $P' \longrightarrow_{\beta 12} P$, we can also derive $(\overline{\lambda}x.P')\ N \hookrightarrow (\overline{\lambda}x.P')\ N' \longrightarrow_{\beta 12} (\overline{\lambda}x.P)\ N'$.

  If $M = (\overline{\lambda}x.P)\ M' \rightsquigarrow_2 (\overline{\lambda}x.P)\ N$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow Q \longrightarrow^{+}_{\beta 12} N'$ and the weak contextual closure gives $(\overline{\lambda}x.P)\ M' \hookrightarrow (\overline{\lambda}x.P)\ Q \longrightarrow^{+}_{\beta 12} (\overline{\lambda}x.P)\ N'$.

$\square$

Lemma 10 $\longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}}$

Proof: By induction on the reduction step $\hookrightarrow$.

- For the first base case, we have $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\lambda x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked subterm. Since root $\overline{\mathsf{assoc}}\mathsf{act}$-reduction produces neither λ-abstractions nor applications at the root, note that M has to be of the form $(\lambda x.N')\ P'$, with either $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$ (and $P' = P$) or $P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} P$ (and $N' = N$). In both cases, $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N')\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\}N' \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} \left\{{}^{P}\!/\!_{x}\right\}N$.

- For the second base case, we have $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked subterm. We do a case analysis on $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P$.

If $M = (\overline{\lambda}x'.M_1)\ (\overline{\lambda}x.M_2)\ P \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}x.(\overline{\lambda}x'.M_1)\ M_2)\ P$ with $N = (\overline{\lambda}x'.M_1)\ M_2$, we also have $M = (\overline{\lambda}x'.M_1)\ (\overline{\lambda}x.M_2)\ P \hookrightarrow (\overline{\lambda}x'.M_1)\ \left\{{}^{P}\!/\!_{x}\right\} M_2 = \left\{{}^{P}\!/\!_{x}\right\} N$.

If $M = (\lambda x.N)\ P \longrightarrow_{\mathsf{act}} (\overline{\lambda}x.N)\ P$ then $M \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\} N$.

If $M = (\overline{\lambda}x.N')\ P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P$ with either $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$ (and $P' = P$) or $P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} P$ (and $N' = N$), we have, in both cases, $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N')\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\} N' \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} \left\{{}^{P}\!/\!_{x}\right\} N$.

- The closure under λ is straightforward.

- For the closure under application, left-hand side, we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N \hookrightarrow M'\ N$ with $M \hookrightarrow M'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$.

  If $Q = M''\ N \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $M'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M$, the induction hypothesis provides $M'' \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} M'$ so $M''\ N \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} M'\ N$.

  If $Q = M\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$, we also have $M\ N' \hookrightarrow M'\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M'\ N$.

- For the closure under application, right-hand side, we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N \hookrightarrow M\ N'$ with $N \hookrightarrow N'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$.

  If $Q = M'\ N \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $M' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M$, we also have $M'\ N \hookrightarrow M'\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N'$.

  If $Q = M\ N'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $N'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$, the induction hypothesis provides $N'' \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} N'$ so $M\ N'' \hookrightarrow \cdot \longrightarrow^{*}_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N'$.

- For the closure under marked redex, the $\hookrightarrow$-reduction can only come from the right-hand side because of the weak contextual closure ($\hookrightarrow$ does not reduce under $\overline{\lambda}$), so we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}y.M)\ P \hookrightarrow (\overline{\lambda}y.M)\ P'$ with $P \hookrightarrow P'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}y.M)\ P$.

  If $Q = (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}y.(\overline{\lambda}x.M')\ N)\ P$ with $M = (\overline{\lambda}x.M')\ N$, we also have $Q = (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P \hookrightarrow (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P' \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}y.(\overline{\lambda}x.M')\ N)\ P'$.

  If $Q = (\lambda y.M)\ P \longrightarrow_{\mathsf{act}} (\overline{\lambda}y.M)\ P$, then we also have $Q = (\lambda y.M)\ P \hookrightarrow (\lambda y.M)\ P' \longrightarrow_{\mathsf{act}} (\overline{\lambda}y.M)\ P'$.

$\square$